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CONES AND VIETORIS-BEGLE TYPE THEOREMS(1)

BY

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ABSTRACT. Infinite cone constructions are exploited to yield diverse generalizations of the Vietoris-Begle theorem for paracompact spaces and Abelian group sheaves. The constructions suggest natural space, map classifications designated as almost p-solid. The methods are extended to upper semicontinuous closed multivalued maps and homotopies and culminate in a disk fixed point theorem for possibly nonacyclic point images.

Introduction. The central position of the Vietoris-Begle theorem in various developments is the impetus for the extensions and generalizations in this paper. Some of the results connect with earlier work by the author and expand and sharpen the methods used. The underlying technique is that of smoothing out certain point antecedents under mappings by introducing covering cones sticking out in diverse directions [1]. These cones are introduced as quotient spaces, but may be developed as subspaces of intermediate spaces. There are difficulties, when too many cones enter, as regards assurance that the space so fortified is still Hausdorff, and that the natural extensions of the maps are continuous. For this reason the author introduced certain space map pairs in [2] and [3] under the designation of almost p-solid. Though the author's earlier results were for compact spaces and an Abelian coefficient group, the results are established here for paracompact spaces (or more generally for paracompactifying support families) and for coefficient sheaves of Abelian groups or modules.

Attention may also be called to the work of Sklyarenko [5]. While his completely different methods yield results overlapping some of those given both here and in the author's earlier cited articles, neither set of such results includes the other. If the initial space is *both* compact and metric, Sklyarenko's results are stronger, but if either restriction is dropped our results may be more powerful, as shown by examples. An understanding of the relation of these differing viewpoints is, perhaps, furnished by the observation that in [5] the

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dimension of the singular sets of the *image* space is central, while in the author's work the associated (ApS) properties of the *domain* space are predominant.

Formulation of analogues of the almost p-solid property, when triples of spaces or relations enter, allows extension of theorems of the type of those occuring in the work of Białynicki-Birula [6], Wallace [10] and Lawson [11].

For upper semicontinuous set valued transformations in paracompact spaces the cone construction is applicable to the image space and the ApS concept is again effective and extends also to set valued transformations and homotopies. A basic cohomology group isomorphism valid in merely a critical range of dimensions is established. A striking consequence is a new type of fixed point theorem where for a restricted situation certain point images under the set valued transformation need not be acyclic. This is a significant advance over the Eilenberg-Montgomery theorem [14] and would seem to initiate a far reaching breakthrough.

1. **Preliminaries and conventions.** Our standing hypotheses are that *our spaces are paracompact*, though most of the results hold if we merely require certain support families to be paracompactifying. When not otherwise stated ϕ and ψ are the families of all closed sets. However a presuperscript dot thus, $\dot{H}(x)$, connotes compact supports. *Almost all* means with at most a "finite number of exceptions". A *map* is understood to be continuous. *Invariably p is a positive integer and ϵ is a positive real.*

Let W and Y be paracompact spaces, but not necessarily distinct. Let S_p be a subset of Y referred to as the *singular set*. Let $W(y)$ be a closed subset of W, but it is not required that $W(y)$ and $W(y')$ be distinct for $y \neq y'$. The typical point of $W(y)$ is $w(y)$.

We shall need the concept of almost p-solid introduced in [2], [3]. Basically this involves a pairing of W and Y through $W(y)$. Thus we refer to W, Y as *almost p-solid*, or, when $W(y)$ is $f^{-1}(y)$, we refer to W, f as *almost p-solid*, etc. Specifically, W, Y is almost p-solid or ApS if for every $w_0 \in W$ there is a neighborhood basis at w_0 such that, for $N(w_0)$ in this basis and for almost all $y \in S_p$, $W(y) \cap N(w_0) \neq \emptyset$ implies $W(y) \subset N(w_0)$. This seems a stronger restriction than that in [2] of [3], which will be referred to as $ApSw$, which required that for $N(w_0)$ there be an included $N'(w_0)$ so that, for almost all $y \in S_p$, $W(y) \cap N'(w_0) \neq \emptyset$ implies $W(y) \subset N(w_0)$. However, we shall show (2.7) that *they are equivalent.* Similar notions in different but related situations are taken up in §10. We introduce the terminology *$f^*(m)$ is a pq morphism* to indicate that $f^*(m)$ is an epimorphism for $m = p$, an isomorphism for $p < m \leq q$ and a monomorphism for $m = q + 1$.

We suppose the sheaf $\mathfrak{B}$ *is a sheaf of Abelian groups on Y*, but we may also take it to be a sheaf of modules over a sheaf of rings. We write $\mathfrak{A}$ for the inverse sheaf $f^*\mathfrak{B}$ and ϕ and ψ for the support families on X and on Y. The key map, f, is *closed* and *surjective* on X onto Y.

Let I be a fixed unit segment. For each $y \in S_p$ let $I(y)$ be the unit segment $0 \le t \le 1$. Let $\Pi = \Pi_{y \in S_p} I(y)$ be the parallelotope whose points are the functions $\psi \in \Pi$ with $\psi(y) \in I$. The topology of Π is the usual Tychonoff topology, so Π is compact T_2. The function identically 0 for all y is denoted by $*$. The function $1_{y_0} \in \Pi$ is defined by $1_{y_0}(y_0) = 1$, $1_{y_0}(y) = 0$ for $y \neq y_0$. The subset $J(y_0)$ consists of the functions $\{t\, 1_{y_0} \mid 0 \le t \le 1\}$.

$$P = W \times \prod_{S_p} I(y).$$

The *cylinder* over $W(y)$ is defined for $y \in S_p$ as

$$B(y) = W(y) \times J(y).$$

The *cylinder set* B is

$$B = \bigcup_{S_p} (W(y) \times J(y)) \cup \bigcup_{y \notin S_p} (W(y) \times *) = \left(\bigcup_{S_p} W(y) \times J(y)\right) \cup W \times *.$$

Identify the *roof* $W(y) \times 1_y$, $y \in S_p$, to a point $w(y)^*$ to get the cone $W(y)^* \equiv W^*(y)$ with *vertex* $w(y)^*$. Then define

$$B^* = \bigcup_{S_p} W(y)^* \cup \bigcup_{y \notin S_p} W(y) \times * = \bigcup_{S_p} W(y)^* \cup W \times *.$$

We often tacitly identify $W \times *$ and W. The topology of B^* is the identification topology. The *identification map* $B \longrightarrow B^*$ *is denoted by* g. We often refer to points of B or of B^* (other than cone vertices) by (w, ψ), but a point of $W(y) \times *$ is often written simply $w(y)$ or w, and neighborhoods in $W \times *$ are written $N(w)$.

2. **Basic lemmas.** Since P is a product of a paracompact space and a compact space, *it is paracompact*. The following two lemmas are crucial.

Lemma 2.1. *If* W, f *is ApS,* B^* *is Hausdorff and paracompact.*

We first establish that B is closed in P and therefore paracompact. Define

$$T(w_0) = \{y \mid w_0 \cap W(y) \neq \emptyset\} \cap S_p.$$

Suppose $(w_0, \psi_0) \notin B$. Then either $\psi_0(y) \neq 0$ for some $y_1 \notin T(w_0)$ or $\psi_0(y) \neq 0$ for a pair y_1, y_2 with $y_1 \in T(w_0)$. In the first case choose the neighborhood

$$V(\psi_0) = \{\psi \mid |\psi(y_1) - \psi_0(y_1)| < \delta/2, \text{ where } \psi_0(y_1) = \delta > 0\}.$$

The second case includes the possibility that $T(w_0) = S_p$. Here if $\psi_0(y_i) = 2\delta_i > 0$, $i = 1, 2$,

$$V(\psi_0) = \{\psi \mid |\psi(y_1) - \psi_0(y_1)| < \delta_1, \ |\psi(y_2) - \psi_0(y_2)| < \delta_2\}.$$

In the first case choose $N(w_0) = W(y_1)^{\sim}$. In the second case choose $N(y_0)$ arbitrarily. Then $N(w_0) \times V(\psi_0) \cap B = \varnothing$. Hence B is closed in P.

We now establish that B^* *is Hausdorff* if W, f is ApS. The required separation property can be in doubt only for points on a common $W(y_0)$. Suppose then w_1 and w_2 are distinct points on $W(y_0)$. Pick disjunct neighborhoods $N(w_1)$ and $N(w_2)$ to satisfy ApS. Let

$$\rho_i = \{y \mid W(y) \cap N(w_i) \neq \varnothing, \ W(y) \cap N(w_i)^{\sim} \neq \varnothing\} \cap S_p, \qquad i = 1, 2,$$

be the finite subsets of S_p. Let

$$U = \{\psi \mid \psi(y) < 1/2, \ y \in \rho_1 \cup \rho_2\}.$$

Then note $g^{-1}g((N(w_i) \times U) \cap B) = (N(w_i) \times U) \cap B$. Hence, $g((N(w_i) \times U) \cap B)$ is open in B^*. Plainly,

$$g((N(w_1) \times U) \cap B) \cap g((N(w_2) \times U) \cap B) = \varnothing.$$

Accordingly w_1 and w_2 can be separated in B^*, whence B^* is Hausdorff.

Next we show g *is a closed map*. Let C be a closed set in B. Define C^+ by the condition that if c is a point in a roof in C, the entire containing roof is to be included in C^+. Our primary assertion is that C^+ is a closed set.

We separate the proof into two cases.

Case 1. Let $b_0 \in B \cap C^{\sim}$ be any point not in $W \times *$ nor on a roof. Thus

$$b_0 = (w_0, s1_{y_0}) \in W(y_0), \qquad y_0 \in S_p, \ 0 < s < 1.$$

Since b_0 can be separated from C by a neighborhood $O(b_0)$ we can assume

$$O(b_0) = (V(w_0) \times U(s1_{y_0})) \cap B$$

where $V(w_0)$ is open in W and $U(s1_{y_0}) = \{\psi \mid s/2 < \psi(y_0) < 1\}$. Since $(w, \psi) = b \in B$ implies $\psi(y)$ can differ from 0 for at most a single y, only points of the type (w, ψ) with $\psi(y) = 0$, $y \neq y_0$ enter in $O(b_0)$. In short, no roof points appear. Hence $O(b_0)$ separates b_0 from C^+.

Case 2. Suppose $b_0 = w_0$, $*$ is disjunct from C. If b_0 could not be separated from C^+, this would imply the existence of a collection of roof points $r_\alpha = w(y_\alpha) \times 1_{y_\alpha} \in C^+$, where $y_\alpha \in S_p$, which admit b_0 as a cluster point. By ApS every neighborhood $N(w_0)$ of a base at w_0 contains $W(y_\alpha)$ if it contains $w(y_\alpha)$ for almost all $y_\alpha \in S_p$. Therefore, for any $U(*)$, $N(w_0) \times U \cap B$ contains $(W(y_\alpha) \times (1_{y_\alpha}))$ with at most a finite number of exceptions. Hence the collection $\{v(y_\alpha) \times 1_{y_\alpha} \in C\}$ where $v(y_\alpha) \in W(y_\alpha)$ must admit the cluster point b_0 also, contrary to our hypothesis that $b_0 \notin C$. It therefore follows that $g^{-1}gC = C^+$ and $g^{-1}gC^+ = C^+$. Hence $gC^+ = gC$ is closed in B^*.

Since g is continuous and closed, and B is paracompact, the image of g, namely B^*, is paracompact.

Corollary 2.2. *If W is compact, B^* is compact Hausdorff.*

With f a surjective and closed map on X to Y, interpret W and $W(y)$ as X and as $f^{-1}(y) = X(y)$ and write X^* for B^*. Extend f to the map F on X^* onto Y by $F(x(y), s1_y) = f(x(y)) = y$, $F(x(y)^*) = y$, $s < 1$.

Lemma 2.3. *F is continuous.*

This follows from the fact that g is the identification map. Thus consider the commutative square

$$\begin{array}{ccc} B & \xrightarrow{g} & X^* \\ \downarrow p & & \downarrow F \\ X & \xrightarrow{f} & Y \end{array} \qquad (2.31)$$

where p projects B onto X (identified with $X \times *$). Then, for every open set U in Y, $F^{-1}U = gp^{-1}f^{-1}U$. Define the open set

$$V = p^{-1}f^{-1}U = \bigcup_{y \in U \cap \tilde{S}_p} W(y) \times * \cup \bigcup_{y \in U \cap S_p} W(y) \times J(y)$$

$$= W \times * \cup \bigcup_{y \in U \cap S_p} W(y) \times J(y).$$

Since a roof is entirely *in* or entirely *out* of V, $g^{-1}gV = V$. Hence gV is open, that is to say $F^{-1}(U)$ is open.

We shall have frequent use for the following elementary deductions.

Corollary 2.4. (a) *$f^{-1}(y)^*$ is closed in B^*.*
(b) *$C \times *$ is closed in B^* if C is closed in W.*

Assertion (a) follows from Lemma 2.3. For (b), C closed in W implies $C \times *$ is closed in B. The fact that g is closed, as established incidentally in Lemma 2.1, finishes the demonstration.

We need the known

Lemma 2.5. *If A and B are closed subsets of $X = A \cup B$ and A and B are paracompact, then X is paracompact.*

(This follows from the closure of the mapping of the disjunct union of A and B onto $A \cup B$.)

A simple and most useful observation is that embodied in

Lemma 2.6. *If* $X = \bigcup U_\alpha$ *where* U_α, $\alpha \in A$, *is open and no two* U_α*'s overlap, then*

$$H^*_\phi(X, \mathfrak{A}) \approx \prod_A H^*_{\phi|U_\alpha}(U_\alpha, \mathfrak{A})$$

where Π *indicates the direct product (sometimes written* Σ*). If* ϕ *is a compact support family the direct product is replaced by the direct sum.*

The essential idea is that the sections of a sheaf over X are independent over the individual U_α's. In more detail, with $\mathfrak{A}_\alpha$ the sheaf conventionally indicated by $\mathfrak{A}_{U_\alpha}$ (namely, $\mathfrak{A} \mid U_\alpha$ on U_α and 0 on $X - U_\alpha$) there is the direct product representation

$$\mathfrak{A} \approx \prod_A \mathfrak{A}_\alpha. \tag{2.6a}$$

(Since every open $V(x)$ contains an open $V'(x)$ in U_α with $V' \cap U_{\alpha'} \neq \emptyset$, for some α' the interpretation of $\mathfrak{A}$ as a direct limit of presheaves indicates the direct sum is equivalent to the direct product here.) Plainly, $\mathcal{C}^0(X, \mathfrak{A}_\alpha)(x) = 0$, $x \notin U_\alpha$, whence

$$(\mathcal{C}^0(X, \mathfrak{A}))_\alpha = \mathcal{C}^0(X, \mathfrak{A}_\alpha). \tag{2.6b}$$

Accordingly, just as for (2.6a),

$$\mathcal{C}^0(X, \mathfrak{A}) \approx \prod_A \mathcal{C}^0(X, \mathfrak{A}_\alpha). \tag{2.6c}$$

Moreover with $\mathcal{Z}^1(X, \mathfrak{A}_\alpha)$ the cokernel of $\mathfrak{A}_\alpha \to C^0(X, \mathfrak{A})$, $\mathcal{Z}^1(X, \mathfrak{A})_\alpha(x) = 0$ for $x \notin U_\alpha$ and hence, as in the case of (2.6b) and (2.6c), $\mathcal{Z}^1(X, \mathfrak{A})_\alpha \approx \mathcal{Z}^1(X, \mathfrak{A}_\alpha)$ and $\mathcal{Z}'(X, \mathfrak{A}) \approx \Pi_A \mathcal{Z}'(X, \mathfrak{A}_\alpha)$. The definition $\mathcal{C}^n(X, \mathfrak{A}) \approx \Pi_A\, \mathcal{C}^n(X, \mathcal{Z}^n(X, \mathfrak{A}_\alpha))$ then yields by induction $\mathcal{C}^n(X, \mathfrak{A}) \approx \Pi_A\, C^n(X, \mathfrak{A}_\alpha)$. Accordingly with Γ_ϕ the module of sections

$$\Gamma_\phi \mathcal{C}^n(X, \mathfrak{A}) \approx \prod_A \Gamma_\phi \mathcal{C}^n(X, \mathfrak{A}_\alpha) \quad \left(\approx \prod_A \Gamma_{\phi|U_\alpha} \mathcal{C}^n(X, \mathfrak{A}) \quad [7, \text{p. } 17]\right).$$

Then by [7, p. 51] there results the assertion of the lemma.

If K is compact, K intersects at most a finite number of U_α's whence the second part of the lemma.

We now demonstrate a significant unifying principle alluded to earlier.

Lemma 2.7. *The* ApS *property is equivalent to the* $ApSw$ *property.*

That ApS implies $ApSw$ is obvious.

Suppose (X, f) is $ApSw$. Then let $N(w_0)$, $N'(w_0)$ be associated neighborhoods satisfying: $W(y) \cap N'(w_0) \neq \emptyset$ implies $W(y) \subset N(w_0)$ for almost all $y \in S_p$. Let

$$M(w_0) = \bigcup \{W(y) \mid W(y) \cap N'(w_0) = \varnothing,\ W(y) \cap N(w_0)^{\sim} \neq \varnothing,\ W(y) \cap N(w_0) \neq \varnothing\}.$$

We assert $C(w_0) = M(w_0) \cap N(w_0)$ is closed in $N(w_0)$. If not, there is a point $w_1 \in (\overline{C}(w_0) \cap C(w_0)^{\sim}) \cap N(w_0)$. Therefore every neighborhood of w_1 meets a nonfinite subcollection of $\{W(y) \mid y \in S_p\}$ and each such $W(y)$ intersects $N(w_0)^{\sim}$. Since $N(w_0)$ is also a neighborhood of w_1, no neighborhood $N'(w_1)$ exists with the $ApSw$ property, in contradiction with our hypothesis. Hence $N_1(w_0) = N(w_0) \cap C(w_0)^{\sim}$ has the ApS property.

For compact spaces, that B^* is T_2 follows from (2.1) and (2.7).

3. **Basic theorems.** Many of the later results make use of the assertions and demonstrations in this section. The singular points are defined by breakdown of acyclicity. Thus for the following theorem,

$$S_p = \{y \mid H^m_{\phi \cap f^{-1}(y)}(f^{-1}(y), \mathfrak{A}) \neq 0 \quad \text{for some } 0 < m < p \text{ or } H^0_{\phi \cap f^{-1}(y)}(f^{-1}(y), \mathfrak{A}) \not\approx \mathfrak{B}_y\}.$$

Since this depends on $\mathfrak{A} = f^*(\mathfrak{B})$ the notation in §2 is amplified to $(X, f, \mathfrak{B})$ *in place of* (X, f).

Theorem 3.1. *If besides our standing hypotheses f is closed, (X, f, B) is ApS, and if $H^m_{\phi \cap X(y)}(X(y), \mathfrak{A}) = 0$, $p \leq m \leq q$, then $f^*(m)\colon H^m_\psi(Y, \mathfrak{B}) \to H^m_\phi(X, \mathfrak{A})$ is a pq morphism.*

We define the extension of f to $F\colon X^* \to Y$ by

$$FX^*(y) = fX(y) = y.$$

Lemma 2.3 guarantees continuity. We assert that F is *closed.* Since $F = fpg^{-1}$ (cf. (2.31)) and $g^{-1}C$ is closed when C is closed, and f is closed, it is sufficient for the demonstration to show p is closed. Thus let B_0 be a closed subset of B and let its projection pB_0 be X_0 (identified with $X_0 \times *$). Suppose X_0 were not closed. Then some $x_0 \notin X_0$ would be a cluster point of X_0, i.e. every neighborhood $N(x_0)$ would contain a nonfinite subcollection of points of X_0. However, each such point x_α is either the projection of $x_\alpha \times s1_y \in B_0$ for some y and s or of $x_\alpha \times *$. Hence almost all these points are contained in any neighborhood $N(x_0) \times U \cap B$ where U is open in Π. Therefore x_0 is a cluster point of B_0, and hence is in B_0, and so $x_0 \in X_0$, a contradiction.

The family ϕ extends to ϕ^* the family of all closed sets on X^* and $\phi^* \cap (X \times *)$ is ϕ (since X is identified with the closed subset $X \times *$ of X^*). Accordingly the notation will be simplified by *utilizing ϕ for both ϕ and ϕ^** with the understanding that ϕ is given the interpretation ϕ^* if the argument of H^* includes either X^* or $X^*(y)$. Moreover, $^*\mathfrak{A} = F^*\mathfrak{B}$ is constant on each $X^*(y)$. Write $^*\mathfrak{A} \mid X^*(y) = {}^*_y\mathfrak{A}$. Since $X(y)^*$ is a cone, this sheaf constancy on $X^*(y)$ implies

$$(3.11)\qquad \Gamma({}^*_y\mathfrak{A}) \approx H^*_\phi(x(y)^*, {}^*_y\mathfrak{A}) \approx H^*_\phi(X(y)^*, {}^*_y\mathfrak{A}) \qquad [7, \text{p. } 56].$$

The exact cohomology sequence for the pair $X^*(y)$, $X(y)$ where $X(y)$ is written for $X(y) \times *$ is

$$\to H^m_{\phi \cap X^*(y)}(X^*(y), {}^*_y\mathfrak{A}) \to H^m_{\phi \cap X(y)}(X(y), \mathfrak{A}) \to H^{m+1}_{\phi \cap X^*(y)}(X^*(y), X(y), {}^*_y\mathfrak{A}) \to .$$

Hence, by (3.11),

$$(3.12)\qquad H^m_{\phi \cap X(y)}(X(y), \mathfrak{A}) \approx H^{m+1}_{\phi \cap X^*(y)}(X^*(y), X(y), {}^*_y\mathfrak{A}), \qquad m > 0.$$

Since $X \times *$ is closed in X^* and ϕ is paracompactifying,

$$(3.13)\qquad H^*_\phi(X^*, X, {}^*\mathfrak{A}) \approx H^*_{\phi|X^*-X}(X^* - X, {}^*\mathfrak{A}) \qquad [7, \text{p. } 62].$$

$X^*_\cdot(y_0) - X(y_0)$ is open since it is $g(U(X(y_0)) \times V \cap B)$ with U open in X and $V = \{\psi |\ 0 < \psi(y_0)\}$ and

$$X^* - X = \bigcup_{y \in S_p} (X^*(y) - X(y)).$$

Hence by Lemma 2.6 (recall ϕ is written for ϕ^*)

$$(3.14)\qquad \begin{aligned} H^{m+1}_{\phi|X^*-X}(X^* - X, {}^*\mathfrak{A}) &\approx \prod_{S_p} H^{m+1}_{\phi|X^*-X}(X^*(y) - X(y), {}^*\mathfrak{A}) \\ &\approx \prod_{S_p} H^{m+1}_{\phi \cap X^*(y)}(X^*(y), X(y), {}^*\mathfrak{A}) \approx \prod_{S_p} H^m_{\phi \cap X(y)}(X(y), \mathfrak{A}). \end{aligned}$$

The key diagram is now

$$(3.15)\qquad \begin{array}{ccccccc} \to H^{m+1}_\phi(X^*, X, {}^*\mathfrak{A}) \to & H^{m+1}_\phi(X^*, {}^*\mathfrak{A}) & \xrightarrow{\alpha(m+1)} & H^{m+1}_{\phi|X}(X, \mathfrak{A}|X) & \to H^{m+2}_\phi(X^*, X, {}^*\mathfrak{A}) \to \\ & \nwarrow F^*(m+1) & & f^*(m) \nearrow & \\ & & H^{m+1}_\psi(Y, \mathfrak{B}) & & \end{array}$$

where the horizontal line is, of course, exact and the triangle is commutative.

For Theorem 3.1 the hypotheses and (3.13) guarantee $H^m_\phi(X^*, X, {}^*\mathfrak{A}) = 0$, $p \leq m \leq q$, $p > 0$. Next, by (3.11), $X^*(y)$ is ϕ-acyclic and $X(y)$, $y \notin S_p$, is ϕ-acyclic through $m = p$. Hence $F^{-1}(y)$ is ϕ-acyclic through $m = p$ for all $y \in Y$. Moreover $F^{-1}(y)$ is ϕ-taut in view of [7, 10.4d, p. 52]. Accordingly, since $X^*(y)$, $y \in S_p$, is connected, the conventional Vietoris-Begle theorem is valid [7, p. 55], in the form:

$F^*(q + 1)$ is a monomorphism and $F^*(m)$ is an isomorphism for $m \leq q$.

Accordingly first set $m = q$ and note that from $f^*(m+1) = \alpha(m+1)F^*(m+1)$ there results $f^*(q+1) = \alpha(q+1)F^*(q+1)$ is a monomorphism since $\alpha(q+1)$ is a monomorphism. Then set $m = p-1$ to derive $f^*(p) = \alpha(p)F^*(p)$ is an epimorphism since here $\alpha(p)$ is an epimorphism and $F^*(p)$ is an isomorphism.

Theorem 3.2. *If, besides our standing hypotheses,* $(X, f, \mathfrak{B})$ *is* $A(q+1)S$ *and* $f^*(m)$ *is a* pq *morphism, then*

$$H^m_{\phi\cap X(y)}(X(y), \mathfrak{A}) = 0, \qquad p \le m \le q.$$

In the triangle (cf. (3.15)) with ϕ written for the extension to X^* also,

$$\begin{array}{ccc} H^m_\phi(X^*, {}^*\mathfrak{A}) & \xrightarrow{\alpha(m)} & H^m_\phi(X, \mathfrak{A}) \\ {\scriptstyle F^*(m)}\nwarrow & & \nearrow{\scriptstyle f^*(m)} \\ & H^m_\psi(Y, \mathfrak{B}) & \end{array}$$

$F^*(m)$ is an isomorphism for $m \le q$ and is a monomorphism for $m = q+1$ [7, p. 55]. Then from $\alpha(m)F^*(m) = f^*(m)$ if $F^*(m)$ and $f^*(m)$ are monomorphisms, so is $\alpha(m)$. This is the case for $m = q+1$. If $f^*(m)$ and $F^*(m)$ are epimorphisms, so is $\alpha(m)$. This is the case for $m = p$. Both situations maintain for $p < m \le q$. Apply these facts to the exact sequence of the horizontal line in (3.15) to derive, in consequence of [3, Lemma 9], $H^{m+1}_\phi(X^*, X, {}^*\mathfrak{A}) = 0$, $p \le m \le q$. The assertion of the theorem now follows from (3.13) and (3.14).

4. **Three spaces theorems.** We continue with two theorems of the type of [3, Theorems 12 and 13]. Later we take up a result closer in type to those in [6], with $k = gf$ where f and g are surjective. For completeness we include short proofs of these two theorems, though they amount to obvious modifications of those for the compact situation treated in [3]. For the present and later the situation is

$$(4.11) \qquad X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Parenthetically we remark that we tacitly use the fact that subsets of ApS map-space pairs are ApS.

Theorem 4.1. *Let* X *and* Y *be paracompact and let* Z *be arbitrary. Let* $\mathfrak{B}$ *be a sheaf of Abelian groups on* Y *and let the collection of all closed subsets constitute the support families* ϕ *and* ψ *on* X *and on* Y. *Let* f *be continuous and closed and let* g *be possibly noncontinuous subject to* $g^{-1}z$ *is paracompact. Suppose* X, f *is* $A(q+1)S$. *If*

$$H^m_{\phi\cap k^{-1}(z)}(k^{-1}z, \mathfrak{A}) \approx H^m_{\psi\cap g^{-1}(z)}(g^{-1}z, \mathfrak{B}), \qquad p \leq m \leq q+1,$$

then $f^*(m)\colon H^m_\psi(Y, \mathfrak{B}) \to H^m_\phi(X, \mathfrak{A})$ *is a pq morphism.*

Both here and in the sequel we shall often use ϕ (or ψ) when subspaces enter, thus ϕ for $\phi \cap k^{-1}(z)$, etc. For fixed z, f maps $k^{-1}z$ onto $g^{-1}z$. Interpret these two spaces as the X and Y of Theorem 3.2 whence, for each $y \in Y(z) = g^{-1}(z)$,

$$H^m_{\phi\cap f^{-1}(y)}(f^{-1}(y), \mathfrak{A}) = 0, \qquad p \leq m \leq q. \tag{4.12}$$

Since this is valid for each $Y(z)$, (4.12) is valid for all y. Hence Theorem 3.1 is applicable.

Theorem 4.2. *With the same assumptions as in Theorem* 4.1 *on* $X, Y, Z, f, g, \phi, \psi, \mathfrak{B}$ *suppose* $f^*(m)\colon H^m_\psi(Y, \mathfrak{B}) \to H^m_\phi(X, \mathfrak{A})$ *is a pq morphism; then*

$$f^*_z(m) = f^*(m) \,|\, H^m_{\psi\cap Y(z)}(Y(z), \mathfrak{B})\colon H^m_{\psi\cap Y(z)}(Y(z), \mathfrak{B}) \to H_{\phi\cap X(z)}(X(z), \mathfrak{A})$$

is a pq morphism.

By Theorem 3.2

$$H^m_{\phi\cap X(y)}(X(y), \mathfrak{A}) = 0, \qquad p \leq m \leq q. \tag{4.13}$$

Then, since $Y \supset Y(z) = g^{-1}(z)$, Theorem 3.1 applies with f restricted to $f^{-1}Y(z)$ to give the assertions of our theorem.

5. **Alternatives.** Since a paracompact space is completely regular a formulation of the cone construction as a subspace of a large enough parallelotope rather than as a quotient space can be carried out. (In a current thesis Mrs. E. Thornton has extended unpublished subspace results of the writer's originally intended for this paper to a general equivalence with our quotient space constructions. In this connection the proof of Lemma 2.7 is like a similar one in her thesis.) Two other aspects of the ApS notions are considered in this section.

Examples can be given where our results are sharper than those in [5] for the case that X is compact but nonmetric or X is metric but noncompact. The example for the first case is given in [3] (where on p. 29, x should be X^0 taken as a cone over A, whose vertex at $t = 1$ has Euclidean neighborhoods and on line 12, $X = S^n \vee X^0$). Below we take up the second case for connected spaces.

Let Y_0 be the cone with vertex $(0, 0, 1)$ and base $B\colon 0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$, $x_3 = 0$. Imbed Y_0 in R^4. Let Y be $Y_0 \cup S^3$ where S^3 is attached to Y_0 at the point $(0, 0, 1, 0)$ only. Assume the Euclidean topology for Y.

Let X_1 have the same points as Y but the topology is different. Thus if a point on the cone is labeled (x_1, x_2, s), define a metric by

$$\begin{aligned} d((x_1, x_2, s), (x_1', x_2', s')) &= 1 - s \quad \text{if } s \leq s', \ (x_1, x_2) \neq (x_1', x_2'), \\ &= |s - s'| \quad \text{if } (x_1, x_2) = (x_1', x_2'). \end{aligned}$$

The attached sphere S^3 is given the usual metric topology. At each point of the base B attach a circle S^1. More specifically attach the circles

$$x_1 = x_{10}, \quad x_2 = x_{20}, \quad (x_3 + 1)^2 + x_4^2 = 1 \tag{5.1a}$$

where (x_{10}, x_{20}) ranges over B. These circles are disjunct. On each circle the Euclidean topology is understood. The distance between any two points on different circles is 1. The distance between a point on the circle attached at $(x_{10}, x_{20}, 0, 0)$ and a point on the cone (x_1, x_2, s) is

$$d((x_{10}, x_{20}, 0, 0), (x_{12}, x_{21}, s)) = d((x_{10}, x_{20}, 0, 0), (x_{10}, x_{20}, s)) + |1 - s|.$$

The space X is the union of X_1 and the circles.

Define the map $f: X \to Y$ as the identity on S^3 and on the cone and each circle of (5.1a) is mapped into its base point $x_{10}, x_{20}, 0, 0$. Evidently the topology of X is finer than that of Y. Accordingly, f is continuous. Moreover, the singular set, S_2, is the base B and hence its dimension is 2. Note, (X, f) is $A2S$. Accordingly, by Theorem 3.1, $f^*(m)$ is an isomorphism for $m \geq 3$. On the other hand, Sklyarenko's results yield $f^*(m)$ is an isomorphism for $m \geq 5$ only.

We shall require the analogue of Lemma 15 of [3].

Lemma 5.2. *For X paracompact and $\dot{X} = f^{-1}(\overline{S}_p)$ compact, ApS is equivalent to the condition that for every vicinity V of the diagonal Δ in $X \times X$, $X(y) \times X(y) \subset V$ for almost all $y \in S_p$.*

Let V be assigned. For each $x \in X$ choose an open set $N(x)$ to satisfy $N(x) \times N(x) \subset V$. Pick $M(x) = N(x) \cap \dot{X}^{\sim}$ for all $x \notin \dot{X}$. Next in view of Lemma 2.7 pick $N'(x)$ to satisfy $ApSw$ for each $x \in \dot{X}$, and identify $M(x)$ and $M'(x)$ for $x \notin X$. By compactness a finite subcover $\{N'(x_k) | \ k = 1, \cdots, K\}$ covers $\dot{X}$. Write $Y_k = \{y | \ X(y) \cap N'(x_k) \neq \emptyset\} \cap S_p$. Then $S_p = \bigcup Y_k$. For *at most a finite subset of* Y_k, $X(y)$ meets $N'(x_k)$ and yet lies outside $N(x_k)$, i.e. $X(y) \times X(y)$ is not in $N(x_k) \times N(x_k)$. Since every $X(y)$, $y \in S_p$, meets some $N'(x_i)$, it follows that, for almost all $y \in S_p$,

$$X(y) \times X(y) \subset \bigcup N(x_k) \times N(x_k) \cup \bigcup_{x \notin \dot{X}} M(x) \times M(x) \subset V.$$

The argument for the reverse implication as given in [3] may be taken over unchanged. This reverse implication is true for X paracompact and no restriction on $f^{-1}(\overline{S}_p)$. However, the direct implication is invalid without restriction of $f^{-1}(\overline{S}_p)$ of the type in the lemma. For example, with Y the nonnegative reals and the planar set $W = \bigcup_n \{(y_n, v_n) \mid y_n = 2^n,\ v_n = n\} \cup Y \times 0$, let f project W onto Y. Then (W, f) is ApS, but does not satisfy the conclusion of Lemma 5.2.

6. **Triples of spaces.** We now modify the restrictions on the triple in (4.11). Specifically f and g are closed continuous surjections. Write $X(z)$ and $Y(z)$ as before and use the same families ϕ and ψ. Let $\mathfrak{D}$ be a sheaf of Abelian groups on Z. Let $S_p = \{z \mid H^m_\psi(Y(z), \mathfrak{B}) \not\approx H^m(x(z), \mathfrak{A})\}$ for some m, $0 \le m \le p$, where $\mathfrak{B} = g^*\mathfrak{D}$ and $\mathfrak{A} = f^*\mathfrak{D}$. Again write $f^*_z(m)$ for the induced homomorphism $H^m_\psi(Y(z), g^*\mathfrak{D}) \to H^m_\phi(X(z), h^*\mathfrak{D})$ where just as earlier ϕ, ψ are interpreted as *extensions* to the families of all closed sets when X^* or Y^* enter in arguments of H.

We shall make use of the following diagram under varying conditions. We shall refer to it as the *ladder* or simply as (L).

$$\text{(L)}\quad \begin{array}{ccccccccc} \to H^m_\psi(Y^*, {}^*\mathfrak{B}) & \to & H^m_\psi(Y, \mathfrak{B}) & \to & H^{m+1}_\psi(Y^*, Y, {}^*\mathfrak{B}) & \to & H^{m+1}_\psi(Y^*, {}^*\mathfrak{B}) \to \\ \downarrow F^*(m) & & \downarrow f^*(m) & & \downarrow Q^*(m) & & \downarrow F^*(m+1) \\ \to H^m_\phi(X^*, {}^*\mathfrak{A}) & \to & H^m_\phi(X, \mathfrak{A}) & \to & H^{m+1}_\phi(X^*, X, {}^*\mathfrak{A}) & \to & H^{m+1}_\phi(X^*, {}^*\mathfrak{A}) \to \end{array}$$

where the rows are exact and the squares are commutative.

We define *almost p-solid pairs or ApS pairs $(X, k, f^*\mathfrak{B}; Y, g, \mathfrak{B})$ by the property that $(X, k, f^*\mathfrak{B})$ and $(Y, g, \mathfrak{B})$ are each (ApS) for the same S_p* or more generally S_p is to be the union of the singular subsets of Z for these two triples.

Theorem 6.1. *Suppose $(X, k, f^*\mathfrak{B})$ and $(Y, g, \mathfrak{B})$ are (ApS) pairs and that $H^m_{\psi \cap Y(z)}(Y(z), \mathfrak{B}) \xrightarrow{f^*(m)} H^m_{\phi \cap X(z)}(X(z), f^*\mathfrak{B})$ is an isomorphism for all z and for $p \le m \le q$ and a monomorphism for $m = q+1$. Then $H^m_\psi(Y, \mathfrak{B}) \xrightarrow{f^*(m)} H_\phi(X, f^*\mathfrak{B})$ is a pq morphism.*

Define

$$X^* = \bigcup_{z \in S_p} X^*(z) \cup \bigcup_{z \notin S_p} X(z) \times *,$$

$$Y^* = \bigcup_{z \in S_p} Y^*(z) \cup \bigcup_{z \notin S_p} Y(z) \times *,$$

with $X^*(z)$, $Y^*(z)$ the cones over $X(z)$, $Y(z)$. By Lemma 2.1, in view of the ApS hypotheses, X^* and Y^* are Hausdorff and paracompact.

For convenience of reference we note the following well-known 4-lemmas connected with

$$\begin{array}{ccccccc} A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

where the rows are exact and the squares are commutative and α_2 is an epimorphism.

Lemma 6.2. *If* (a) α_3 *and* α_5 *are monomorphisms then* α_4 *is a monomorphism. If* (b) α_4 *is an epimorphism and* α_5 *is an monomorphism then* α_3 *is an epimorphism.*

We are ready for the proof of Theorem 6.1. Extend g to $G: Y^* \to Z$ by $G(Y^*(z)) = gY(z) = z$. Hence by Lemma 2.3, G is continuous. Similarly $(gf) = k$ extends to $K: X^* \to Z$ by

$$KX(z)^* = g/X(z) = z, \qquad z \in S_p,$$

$$K(X(z) \times *) = g/X(z) = z,$$

and again K is continuous by Lemma 2.3.

Next for each $(x, s1_z) \in X(z)^*, x \in f^{-1}g^{-1}(z), z \in S_p, s < 1$, define $F(x, s1_z) = (f(x), s1_z)$. Evidently, $f(x) \in Y(z)$. Then $F(x, *) = (f(x), *)$ and for the vertex $x(z)^*$, $F(x(z)^*) = (fx(z))^*$. In short, f maps X^* onto Y^*.

Let B' be the B for $Y \xrightarrow{g} Z$ and let B refer to $X \xrightarrow{k} Z$. Then the corresponding identification maps on B' to Y^* and on B to X^* are indicated by γ' and by γ respectively. The map induced by f on B to B' is $f \times i$ where i is the identity map taking ψ into ψ. That $f \times i$ is continuous and closed follows easily from these properties for f and of course for i.

The continuity and closure of F then follow directly. Thus consider the commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\gamma} & X^* \\ f\times i \downarrow & & \downarrow F \\ B' & \xrightarrow{\gamma'} & Y^* \end{array}$$

If O is open in Y^*, $(f \times i)^{-1}\gamma'^{-1}O$ is open in B by continuity. Since the antecedent of a vertex in Y^* is a roof, $\gamma'^{-1}O$ contains only full roofs and hence this is the case for $(f \times i)^{-1}\gamma'^{-1}O$ also. Accordingly, $\gamma^{-1}\gamma$ is the identity on this set and, therefore, $F^{-1}O = \gamma(f \times i)^{-1}\gamma'^{-1}O$ is an open set.

To establish that F is closed we start with a closed subset K of X^*. Then formally $FK = \gamma'(f \times i)\gamma^{-1}K$. By Lemma 2.1, γ' is closed. Since f and i are closed so is $f \times i$. Moreover, y is continuous. Hence FK is closed.

With these properties of F, a special case of [5] (or by [6] when our spaces are compact), yields $F^*(m)$ is an isomorphism for all m. The argument based on Lemma 2.6 yields in analogy to (3.14):

$$H_\psi^{m+1}(Y^*, Y, {}^*\mathfrak{B}) = \prod H^m_{\psi \cap Y(z)}(Y(z), \mathfrak{B})$$

$$H_\phi^{m+1}(X^*, X, F^*\mathfrak{D}) = \prod H^m_{\phi \cap X(z)}(X(z), \mathfrak{A})$$

whence by the hypotheses of the theorem $Q^*(m)$ is an isomorphism for $p \leq m \leq q$ and a monomorphism for $m = q + 1$.

We take up first the special case that $q = p$. Then in the ladder, $F^*(p)$ and $Q^*(p)$ are isomorphisms while $F^*(p+1)$ and $Q^*(p+1)$ are monomorphisms. The first four columns of the ladder then yield, by Lemma 6.2(b), that $F^*(p)$ is an epimorphism. The last four columns yield, by Lemma 6.2(a), that $f^*(p+1)$ is a monomorphism. The general case is then clear.

7. **Pairs.** Extensions to pairs of spaces imply results for possibly non-paracompact spaces, and locally compact spaces extending [15], though such results may be obtainable directly also from our earlier theorems. In this section X_0 and Y_0 are closed subsets and f is a *closed map of* X, X_0 *onto* Y, Y_0. Denote $f \mid X_0$ by f_0. As usual $\mathfrak{B}$ is a sheaf on Y and ϕ, ψ refer to all closed sets.

Theorem 7.1. *For paracompact spaces suppose* $X, f, \mathfrak{A}$ *is ApS. Suppose also* $H^m_{\phi \cap X(y)}(X(y), \mathfrak{A}) = 0$, $p \leq m \leq q$. *Let* $H^m_{\phi \cap X(y_0)}(X(y_0), f_0^*\mathfrak{B}) = 0$ *where* $y_0 \in Y_0$ *and* $0 < p - 1 \leq m \leq q - 1$. *Then*

$$Q^*(m): H^m_\psi(Y, Y_0, \mathfrak{B}) \to H^m_\phi(X, X_0, \mathfrak{A})$$

or equivalently $H^m_{\psi \mid Y - Y_0}(Y - Y_0, \mathfrak{B}) \to H^m_{\phi \mid X - X_0}(X - X_0, \mathfrak{A})$ *is a pq morphism.*

Since X_0 is closed in X, and ϕ is paracompactifying, it is ϕ-taut in X, whence

(7.11) $$H^m_{\phi \mid X - X_0}(X - X_0, \mathfrak{A}) \approx H^m_\phi(X, X_0, \mathfrak{A}) \qquad [7, 2{:}10.4d].$$

Similarly, $H^m_{\psi \mid Y - Y_0}(Y - Y_0, \mathfrak{B}) \approx H^m_\psi(Y, Y_0, \mathfrak{B})$. There results the ladder

(7.12)
$$\begin{array}{ccccccccc}
H^m_\psi(Y, Y_0, \mathfrak{B}) & \to & H^m_\psi(Y, \mathfrak{B}) & \to & H^m_{\psi \cap Y_0}(Y_0, \mathfrak{B}) & \to & H^{m+1}_\psi(Y, Y_0, \mathfrak{B}) & \to & H^{m+1}_\psi(Y, \mathfrak{B}) \\
\downarrow Q^*(m) & & \downarrow f^*(m) & & \downarrow f_0^*(m) & & \downarrow Q^*(m+1) & & \downarrow f^*(m+1) \\
H^m_\phi(X, X_0, \mathfrak{A}) & \to & H^m_\phi(X, \mathfrak{A}) & \to & H^m_{\phi \cap X_0}(X_0, \mathfrak{A}) & \to & H^{m+1}_\phi(X, X_0, \mathfrak{A}) & \to & H^{m+1}_\phi(X, \mathfrak{A})
\end{array}$$

Evidently X_0, f_0 is ApS and, in view of the hypotheses and Theorem 3.1, $f_0^*(p-1)$ is an epimorphism, $f_0^*(q)$ is a monomorphism, and in between $f_0^*(m)$ is an isomorphism. Similarly, $f^*(p)$ is an epimorphism. The conclusion now follows from Lemma 6.2.

Corollary 7.2. *Theorem* 7.1 *is valid for compact spaces.*

Corollary 7.3. *For X and Y locally compact let $(X, f, \mathfrak{B})$ be ApS assuming compact supports, and suppose ${}^{\cdot}H^m(X(y), \mathfrak{A}) = 0$, $p \le m \le q$. Then $f^*(m)$ is a pq morphism.*

Let $X^+ = X \cup x_0$ and $Y^+ = Y \cup y_0$ be the 1-point compactifications of X and Y. Extend f to f^+ where $f^+x_0 = y_0$. Extend $\mathfrak{B}$ to $\mathfrak{B}^+$ on Y^+ by setting $\mathfrak{B}^+_{y_0} = 0$. Similarly extend $\mathfrak{A}$ to $\mathfrak{A}^+$ with $\mathfrak{A}^+_{x_0} = 0$. This is consistent since $f^{-1}y_0 = x_0$. Hence Corollary 7.2 yields the assertions of this corollary for ${}^{\cdot}H^*(X^+, x_0, \mathfrak{A}^+) \to {}^{\cdot}H^*(Y^+, y_0, \mathfrak{B}^+)$. Then apply (7.11) for the desired conclusions.

Corollary 7.4. *For X and Y locally compact and all closed sets as the families of support, suppose $X - X_0$, f is ApS and suppose $H^m_\Phi(X(y), \mathfrak{A}) = 0$ for $y \in Y - Y_0$ with $p \le m \le q$. Suppose $f_0^*(m)$: $H^m_{\psi \cap Y_0}(Y_0, \mathfrak{B}) \to H^m_{\Phi \cap X_0}(X_0, \mathfrak{A})$ is a pq morphism. Then $f^*(m)$: $H^m_\psi(Y, \mathfrak{B}) \to H^m_\Phi(X, \mathfrak{A})$ is a pq morphism.*

We prove this for the case $p = q$. Note first the one point compactifications $X^+ = X \cup x_0$, $X_0^+ = X_0 \cup x_0$, $Y^+ = Y \cup y_0$, $Y_0^+ = Y_0 \cup y_0$ and the sheaf extension to $\mathfrak{B}^+$ and $\mathfrak{A}^+$ as in the last corollary.

Then using (7.11) for the vertical isomorphism

$$\begin{array}{ccccc} & & {}^{\cdot}H^{n+1}(X^+, x_0, \mathfrak{A}^+) & & \\ & \nearrow & \big\downarrow \approx & \searrow & \\ 0 = {}^{\cdot}H^n(x_0, \mathfrak{A}^+) & & & & {}^{\cdot}H^{n+1}(X^+, \mathfrak{A}^+) \to {}^{\cdot}H^{n+1}(x_0, \mathfrak{A}^+) = 0. \\ & \searrow & & \nearrow & \\ & & H^{n+1}_\Phi(X, \mathfrak{A}) & & \end{array}$$

Hence

$$ {}^{\cdot}H^{n+1}(X^+, x_0, \mathfrak{A}^+) \approx H^{n+1}_\Phi(X, \mathfrak{A}) \approx {}^{\cdot}H^{n+1}(X^+, \mathfrak{A}^+). \tag{7.41}$$

Since $X - X_0$ is locally compact, (7.41) is valid in the form

$$\begin{aligned} {}^{\cdot}H^{n+1}((X - X_0)^+, x_0, \mathfrak{A}) &\approx {}^{\cdot}H^{n+1}((X - X_0)^+, \mathfrak{A}^+) \approx H^{n+1}_{\Phi|X - X_0}(X - X_0, \mathfrak{A}) \\ &\approx {}^{\cdot}H^{n+1}(X, X_0, \mathfrak{A}) \qquad [7, 2{:}10.4d]. \end{aligned} \tag{7.42}$$

Since X_0 and X_0^+ are certainly taut in the compact X^+ for a compact support family, the groups in (7.42) are isomorphic to ${}^{\cdot}H^{n+1}(X^+, X_0^+, \mathfrak{A}^+)$ [7, 2:12.9].

Replace $Q^*(m+1)$ by $F^*(m)$ in (7.12). Since

$$H^*_\psi(Y,\ Y_0,\ \mathfrak{B}) \approx \dot{H}^*_\psi(Y^+,\ Y_0^+,\ \mathfrak{B}^+) \approx H^*_{\psi|Y-Y_0}(Y-Y_0,\ \mathfrak{B}),$$

interpret $X - X_0$, $Y - Y_0$ as X and Y of Corollary 7.3 so $F^*(p)$ is an epimorphism and $F^*(p+1)$ is a monomorphism. Since also $f_0^*(p)$ is an epimorphism and $f_0^*(p+1)$ a monomorphism, Lemma 6.2 implies the result for $q = p$. The more general result for $q > p$ is an immediate consequence.

8. Betti numbers.

Lemma 8.1. *Suppose that $A, \cdots, D'$ are Abelian groups, and that the two horizontal lines are exact:*

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D \\
 & & \downarrow{\scriptstyle \lambda} & & & & \\
A' & \longrightarrow & B' & \xrightarrow{\beta'} & C' & \longrightarrow & D'
\end{array}$$

(a) *If A, C and D' are finitely generated and λ is an epimorphism, then C' is finitely generated.*

(b) *If A', C' and D are finitely generated and λ is a monomorphism, then C is finitely generated.*

For (a): Exactness assures B is finitely generated, whence using the hypothesis on λ, B' is finitely generated, and exactness implies the same for C'. Case (b) is essentially the dual.

Various special consequences of Lemma 8.1 can be phrased in reference to Betti numbers by starting with the typical ladder for the pairs X, X_0 and Y, Y_0 and interpreting λ as $f^*(m)$, $f_0^*(m)$ or $F^*(m)$. (We tacitly assume for the remainder of this section that all spaces are paracompact.) As an instance of the application of Lemma 8.1:

Corollary 8.2. *Let X_0 and Y_0 be closed subsets of X and Y.*

(a) *Suppose f is a closed surjection $X, X_0 \to Y, Y_0$. Let $X - X_0$, f be ApS and $H^p_{\phi\cap X(y)}(X(y), \mathfrak{A}) = 0$, and suppose $H^{p+1}_{\psi\cap Y_0}(Y_0, \mathfrak{B})$, $H^p_{\phi\cap X_0}(X_0, \mathfrak{A})$ and $H^{p+1}_\phi(X, \mathfrak{A})$ are finitely generated. Then $H^{p+1}_\psi(Y, \mathfrak{B})$ is finitely generated.*

(b) *Let $(X_0, f|X_0)$ be ApS and suppose $H^p_{\phi\cap X(y)}(X(y), \mathfrak{A}) = 0$ for $y \in Y_0$. Suppose $H^{p+1}_\phi(X, \mathfrak{A})$, $H^{p+1}_\phi(X, X_0, \mathfrak{A})$ and $H^{p+2}_\psi(Y, \mathfrak{B})$ are finitely generated. Then $H^{p+2}_\phi(X, \mathfrak{A})$ is finitely generated.*

Use (3.1) and (7.1). Then 8.2(a) follows from 8.1(b) when $\lambda(m) = F^*(m)$ and 8.2(b) comes from 8.1(a) with $\lambda(m) = f(m)^*$.

No pq hypotheses are invoked in the next corollaries, but our usual hypotheses maintain namely that X and Y are paracompact, ϕ and ψ refer to all closed sets and that $\mathfrak{A} = f^*\mathfrak{B}$ where $\mathfrak{B}$ is a sheaf of Abelian groups.

Corollary 8.3. *Let X, f be ApS where S_p is finite and suppose $H^p_\psi(Y, \mathfrak{B})$ and $H^p_{\phi\cap X(y)}(f^{-1}(y), \mathfrak{A})$ are finitely generated for $y \in S_p$. Then $H^p_\phi(X, \mathfrak{A})$ is finitely generated.*

We refer to (3.15). By the usual Vietoris-Begle theorem $F(p)$ is an epimorphism whence $H^p_\phi(X^*, A)$ is finitely generated. The hypotheses imply that $\Pi_{S_p} H^p_{\phi\cap X(y)}(f^{-1}(y), \mathfrak{A})$ is finitely generated. By exactness of the horizontal sequence in (3.15), $H^p_\phi(X, \mathfrak{A})$ is therefore finitely generated.

Corollary 8.4. *Let X, f be ApS where S_p is finite and suppose $\lambda^*(m)$ is trivial for $m = p > 1$ where $\psi^*(m)$: $H^m_\phi(X, \mathfrak{A}) \to H^m_{\phi\cap f^{-1}(S_p)}(f^{-1}(S_p), \mathfrak{A})$. Then $f^*(p)$ is an epimorphism. If X, f is ApS and $\lambda^*(p-1)$ is trivial also, then $f^*(p)$ is an isomorphism.*

Consider

$$\begin{array}{ccccccc}
\to H^{m-1}_{\phi\cap f^{-1}(S_p)}(f^{-1}(S_p), \mathfrak{A}) & \xrightarrow{d} & H^m_\phi(X, f^{-1}(S_p), \mathfrak{A}) & \xrightarrow{i^*(m)} & H^m_\phi(X, \mathfrak{A}) & \xrightarrow{\psi^*(m)} & H^m_{\phi\cap f^{-1}(S_p)}(f^{-1}(S_p), \mathfrak{A}) \\
 & & \uparrow{\scriptstyle g^*(m)} & & \uparrow{\scriptstyle f^*(m)} & & \\
H^{m-1}_{\psi\cap S_p}(S_p, \mathfrak{B}) & \longrightarrow & H^m_\psi(Y, S_p, \mathfrak{B}) & \xrightarrow{j^*(m)} & H^m_\psi(Y, \mathfrak{B}) & \longrightarrow & H^m_{\psi\cap S_p}(S_p, \mathfrak{B})
\end{array}$$

For $m > 1$, $j^*(m)$ is plainly an isomorphism since S_p is finite. The hypothesis implies $i^*(p)$ is an epimorphism. By excision [7, p. 62],

$$H^m_\phi(X, f^{-1}(S_p), \mathfrak{A}) \approx H^m_{\phi|X-f^{-1}(S_p)}(X - f^{-1}(S_p), \mathfrak{A}).$$

Since there are no singular points in dimensions through $p-1$ in the mapping $X - f^{-1}(S_p) \to Y - S_p$ and since the support families are paracompactifying, the standard Vietoris-Begle theorem [7, p. 142] applies to yield $g^*(m)$ is an epimorphism for $m = p$. Hence by the commutativity of the square $f^*(p)$ is an epimorphism. If $\lambda^*(p-1)$ is trivial also, then by exactness $i^*(p)$ is a monomorphism. Thus $i^*(p)$ is an isomorphism. Since with the $A(p+1)S$ property, $g^*(p)$ is an isomorphism, it then follows that $f^*(p)$ is an isomorphism.

Corollary 8.5. *If X, f is $A(p+3)S$, and if $H^p_\phi(X, \mathfrak{A})$ and $H^{p+2}_\psi(Y, \mathfrak{B})$ are of finite type, and if $f^*(p+1)$ is an isomorphism, then with $\Sigma_m = \{y \mid H^m_{\phi\cap f^{-1}(y)}(f^{-1}(y), \mathfrak{A}) \neq 0\}$, Σ_p and Σ_{p+1} are finite and $H^m_{\phi\cap f^{-1}(y)}(f^{-1}(y), \mathfrak{A})$ is of finite type for $y \in S_{p+1}$ and $m = p$ and $p+1$.*

Since $f^*(p+1)$ is an isomorphism, so also is $\alpha^*(p+1)$ in (3.15). If

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \xrightarrow{\epsilon} F$$

is an exact sequence of modules, then γ an isomorphism implies α is an epimorphism and ϵ is a monomorphism. We apply this simple observation to the horizontal line of (3.15) when (omitting the sheaves)

$$H^p_\Phi(X, \mathfrak{A}) \to H^{p+1}_\Phi(X^*, X, {}^*\mathfrak{A}) \approx \prod_{S_p} H^p_{\Phi \cap f^{-1}(Y)}(f^{-1}(y), \mathfrak{A})$$

is an epimorphism and hence $\Pi_{S_{p+1}} H_{\Phi\cap f^{-1}(y)}(f^{-1}(y), \mathfrak{A})$ is of finite type so Σ_p is of finite type.

Similarly, in reference to the argument for Theorem 3.1,

$$\prod_{S_p} H^{p+1}_{\Phi\cap f^{-1}(y)}(f^{-1}(y), \mathfrak{A}) \approx H^{p+2}_\Phi(X^*, X, \mathfrak{A}) \to H^{p+2}_\Phi(X^*, {}^*A) \approx H^{p+2}_\psi(Y, \mathfrak{B})$$

is a monomorphism and therefore $\Pi_{S_{p+1}} H^{p+1}_{\Phi\cap f^{-1}(y)}(f^{-1}(Y), \mathfrak{A})$ is of finite type.

9. **Relations.** We now generalize a theorem essentially due to Wallace [10] and sharpened by Lawson [11]. First some definitions. Let R be a closed set in $Y \times Y$ referred to as a *relation*. Let $L(A) = \{y' \mid (y', y) \in R, y \in A\}$. Define Y, R as *almost p-solid ApS* by substituting in the usual definition of almost p-solid, $L(y)$ for $X(y)$. Let f map X onto Y. Define X, f, R as *almost p-solid, ApS*, in the usual way, but using now for $X(y)$ the interpretation $X(y) = f^{-1}L(y)$.

Theorem 9.1. *Let R be a relation for Y. Let f map the compact set X onto Y. For arbitrary closed sets A and B there is to exist a closed set C such that $L(A) \cap L(B) = L(C)$. The singular set S_p is defined as $\{y \mid \check{H}^m(f^{-1}L(y)) \not\approx \check{H}^m(L(y))$ for some $m < p\}$ where cohomology is reduced Alexander cohomology over a fixed Abelian group. Let Y, R and X, f, R be ApS for this S_p. Suppose $f^*_y(m)$: $\check{H}^m(L(y)) \to \check{H}^m(X(y))$ is an isomorphism for $m = p$ and a monomorphism for $m = p+1$. Then $f^*(m)$: $\check{H}(Y) \to \check{H}^m(X)$ is a monomorphism for $m = p+1$ and an epimorphism for $m = p$. More generally, if $f^*_y(m)$ is an isomorphism for $p \leq m \leq q$ and a monomorphism for $m = q+1$, then $f^*(m)$ is a pq morphism.*

Before entering on the proof a few observations and a lemma are in order. The developments involving $X(y)$ and $L(y)$ are essentially covered in §§1 and 2 with the identification of $W(y)$ as $L(y)$ or as $X(y) = f^{-1}L(y) = \bigcup_{y' \in L(y)} f^{-1}(y')$. We note that $y_1 \neq y_2$ does not imply $L(y_1) \cap L(y_2) = \varnothing$, but this is allowed for in the lemmas of §2 (in contradistinction to [3, Lemmas 2 and 4]).

Lemma 9.2. *X^* and Y^* are compact, T_2, if X and Y are compact.*

Since both X and Y are compact, if W is either X or Y the set P is plainly compact. By Lemma 5.3(a), X^* and Y^* are compact Hausdorff.

Now to the theorem. First the map f is extended to a map $F: X^* \to L^*$ by $FX(y) = f^{-1}X(y)$. F on $X^*(y_0)$ is defined by noting a nonvertex point on $X^*(y_0)$ has the coordinates $(x, s1_{y_0})$.

$$F(x, s1_{y_0}) = (f(x), s1_{y_0}) \quad \text{for } x \in X(y_0),\ y_0 \in S_p.$$

For the vertices $x(y_0)^*$, and $y(y_0)^*$ (the vertex of $L(y_0)^*$), $F(x(y_0)^*) = y(y_0)^*$.

We show that F *is continuous*. Write L for the cylinder space B with $W = Y$ and denote the corresponding B^* by L^*. Retain B when the cylinders are over $W = X$. Let $h: B \to L$ be defined by $h(x, \psi) = f(x), \psi$. Evidently $X(y_0) = \bigcup_{y \in L(y)} f^{-1}(y)$ so $FX(y_0) = L(y_0)$. The correspondence induced on B onto L by f is $f \times i$, where i is the identity map taking ψ into ψ. This correspondence is continuous since, for $\overline{y}, s1_{y_0}$ the neighborhood $(M(\overline{y}) \times V(s1_{y_0})) \cap L$ where $V(s1_{y_0})$ is open in $\Pi_{S_p} 1(y)$ has as antecedent under $f \times i$ the open set $(f^{-1}M(y) \times V(s1_{y_0})) \cap B$. Evidently, $\gamma(f \times i) = Fg$ on B to L^*. Then for an open set $O \subset L^*$,

$$F^{-1}O = g(f \times i)^{-1}\gamma^{-1}O. \tag{9.11}$$

The continuity of γ and of $(f \times i)$ implies $(f \times i)^{-1}\gamma^{-1}O$ is open. Moreover, peaks in O become roofs on L and then on B. That is to say $(f \times i)^{-1}\gamma^{-1}O$ contains full roofs only and so is unchanged by $g^{-1}g$. Hence the right-hand side of (9.11) is open.

We can now finish up the proof of the theorem. Denote by $L'(y)$ either $L^*(y)$ for $y \in S_p$ or $L(y) \times *$ for $y \notin S_p$. Then for A and B there exists a set C for which

$$L'(A) \cap L'(B) = L'(C), \tag{9.12}$$

where $L'(A) = \bigcup_A L'(y)$. The ladder required is

$$\begin{array}{ccccccc}
\longrightarrow \dot{H}^m(L^*) & \longrightarrow & \dot{H}^m(L) & \longrightarrow & \dot{H}^{m+1}(L^*, L) & \longrightarrow \\
\Big\downarrow F^*(m) & & \Big\downarrow f^*(m) & & \Big\downarrow Q^*(m) & \\
\longrightarrow \dot{H}^m(X^*) & \longrightarrow & \dot{H}^m(X) & \longrightarrow & \dot{H}^{m+1}(X^*, X) & \longrightarrow
\end{array}$$

We need the analogues of (3.13) and (3.14) which reduce in our case to $\dot{H}^*(L^*, L) \approx \dot{H}^*(L^* - L)$ and since a compact support set on $L^* - L$ can meet only a finite number of $\{L^*(y) - L(y) \mid y \in S_p\}$, $\dot{H}^{m+1}(L^* - L) \approx \bigoplus_{S_p} \dot{H}^m(L(y))$. Furthermore, $F^*(m)$ is an isomorphism for $0 \leq m \leq q$ and a monomorphism for $q + 1$ by (9.12) and [11]. If $q > p$, then $F^*(p)$, $F^*(p+1)$, $Q^*(p)$ are isomorphisms whence $f^*(p)$ is an epimorphism by Lemma 6.2(b). Similarly, $Q^*(q)$ is an isomorphism while $F^*(q+1)$ and $Q^*(q+1)$ are monomorphisms, so $f^*(q+1)$

is a monomorphism by Lemma 6.2(a). The proof that $f^*(m)$, $p \le m \le q$, is an isomorphism is immediate using Lemma 6.2.

10. Set-valued maps. The results above bear also on multiple-valued (mv) transformations of *points to closed sets.* We restrict ourselves to paracompact *locally compact Hausdorff spaces* in this section. An upper semicontinuous multiple-valued transformation is abbreviated to usc *transformation* which since our spaces are completely regular has the important property that the graph is closed. We require besides that compact sets transform to compact sets. Acyclicity in this section is with respect to reduced Alexander cohomology with Q the coefficient field and closed support family ϕ, omitted when understood.

How an induced homomorphism or a family of induced homomorphisms of the homology rings arises from a set valued map has been given various interpretations. In general no such homomorphisms are defined. One formulation depends on the factorization of f as

$$X \xleftarrow{p_1} \Gamma(f) \xrightarrow{p_2} Y$$

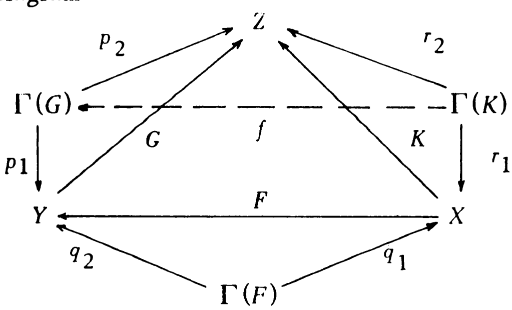
so that $f = p_2 p_1^{-1}$. If $f(x)$ is acyclic for each x, $p_1^{-1}(x) = x \times f(x)$ and the usual Vietoris-Begle theorem yields an isomorphism for p_1^* whence f^* can be *defined* as

$$f^* = p_1^{*-1} p_2^*. \tag{10a}$$

One generalization achieved in the present section is that of defining $f^*(m)$ for those m values for which our earlier results guarantee $p_1^*(m)$ is an isomorphism. It is left open whether or not meaning can be given $f^*(m)$ for other m values.

Lemma 10.1. *Let F, G and K be usc transformations in $X \xrightarrow{F} Y \xrightarrow{G} Z$ where $K = GF$. Let $\Gamma(F)$, $\Gamma(G)$ and $\Gamma(K)$ be the corresponding graphs. Suppose $p_1: \Gamma(G) \to Y$, $r_1: \Gamma(K) \to X$ and $q_1: \Gamma(F) \to X$ are the projections and assume $p_1^*(m)$, $r_1^*(m)$, $q_1^*(m)$ are isomorphisms in the range $p < m \le q$. Then $k^*(m) = F^*(m)G^*(m)$.*

Consider the diagram



All triangles and the hexagon are commutative. This is to say $F = q_2 q_1^{-1}$, $K = r_2 r_1^{-1}$, $G = p_2 p_1^{-1}$. Under the hypotheses of the lemma it follows from (10a) that

$$F^*(m) = q_1^*(m)^{-1} q_2^*(m), \quad K^*(m) = r_1^*(m)^{-1} r_2^*(m), \quad G^*(m) = p_1^*(m)^{-1} p_2^*(m).$$

Finally,

$$K^*(m) = (r_1^{*-1} r_2^*)(m) = (r_1^{*-1} r_1^* q_1^{*-1} q_2^* p_1^{*-1} p_2^*)(m)$$

$$= (q_1^{*-1} q_2^* p_1^{*-1} p_2^*)(m) = F^*(m) G^*(m), \qquad p < m \leq q.$$

Definition 10.2. Let f be a usc transformation on $X \to Y$. Let the singular set be defined by $S_p = \{x \mid f(x)$ is not acyclic for some $m < p\}$. Then X, f is *mv* weakly almost p-solid if for arbitrary $N(y)$ there is an $N'(y) \subset N(y)$ such that, for almost all x in S_p, $f(x) \cap N'(y) \neq \varnothing$ implies $f(x) \subset N(y)$. By an argument like that in Lemma 2.7 one shows that $N'(y)$ can be identified with $N(y)$ and this characterizes the *almost p-solid* or *ApS property.*

A useful form of Lemma 10.1 arises when f and g are usc maps of X onto Y and $r\colon \Gamma(f) \to \Gamma(g)$ is continuous where $r(x, y) = (x, r_x(y))$ for $y \in f(x)$.

Lemma 10.3. *Suppose X, f and X, g are each ApS for the same S_p. Let $r_x^*(m)$, where $H^m(g(x)) \xrightarrow{r_x^*(m)} H^m(f(m))$, be an isomorphism for all $x \in S_p$ and $p \leq m \leq q$ and a monomorphism for $m = q + 1$. Then $f^*(m)$ and $g^*(m)$ exist and are the same for $p \leq m \leq q$ up to an isomorphism.*

Write $p_2 p_1^{-1} = f$ and $q_2 q_1^{-1} = g$ with p_i and q_i the obvious projections. The hypotheses of Theorem 6.1 are easily seen to be satisfied and hence $r^*(m)\colon H^m(\Gamma(g)) \to H^m(\Gamma(f))$ is a pq morphism. Then

$$p_2^*(m) = r^*(m) q_2^*(m), \qquad p \leq m \leq q.$$

$$f^*(m) = p_1^{*-1}(m) p_2^*(m) = p_1^{*-1} r^*(m) q_2^*(m), \quad g^*(m) = q_1^{*-1}(m) q_2^*(m).$$

Since $q_1^*(m)$ and $p_1^{*-1}(m) r^*(m)$ are isomorphisms for $p \leq m \leq q$, the assertion of the lemma stands established.

We turn now to the homotopy notion of relevance to our viewpoint.

Definition 10.4. Let $\mathrm{N}(y_0, I) = \{N(y_0, I)\}$ be a basis of open sets in $Y \times I$ containing $y_0 \times I$, where $N(y_0, I) = \bigcup_{s \in I} M(y_0, s)$ with $M(y_0, s)$ open in $Y \times s$. Let P_s project $Y \times s$ onto Y and write $N(y_0, s)$ for the open set in Y, $P_s M(y_0, s)$. We say $p_1(h)$ is ApS (h is ApS would perhaps be more natural) if for some $N(y_0, I)$ except for a finite subset π of S_p, independent of $s \in I$, $h(x, s) \cap M(y_0, s) \neq \varnothing$ implies $h(x, s) \subset M(y_0, s)$. An apparantly weaker condition denoted by $ApSw$ requires that for every open set $N(y_0)$ in Y there be an open set $N'(y_0)$ such that with a finite number of exceptions $h(x, s) \cap N'(y_0) = \emptyset$, $x \in S_p$, implies $h(x, s) \subset N(y_0)$. We shall show $ApSw$ and ApS are equivalent.

We then say f is *pq homotopic to* g and write $f \sim_{pq} g$ if $h(\ ,s)$, $s \in I$, is acyclic for $p \leq m \leq q$ where $f = h(\ ,0)$ and $g = h(\ ,1)$ and $p_1(h)$ is ApS, while $f \sim_p g$ is used if $q = \infty$.

Definition 10.5. Let h be a pq homotopy on $X \times I \to Y$. The notations Π, 1_x, $J(x)$ are as in §1. Let $\Gamma(x, s) = (x, s, h(x, s))$. In $X \times I \times Y \times \Pi_{S_p}$ construct the cylinders

$$B(x, s) = \Gamma(x, s) \times J(x), \qquad x \in S_p,$$

$$B(h) = \bigcup_{x \in S_p,\ s \in I} B(x, s) \cup \Gamma(h) \times *.$$

Identify each roof $R(x, s) = \Gamma(x, s) \times 1_x$ to a point $v(x, s)$, $x \in S_p$. This defines $\Gamma(h)^*$, i.e. $gB(h) = \Gamma(h)^*$.

We gather together some facts basic to our developments.

Lemma 10.6. (a) *ApS is equivalent to $ApSw$ for $p_1(h)$.*

(b) *Let h be a pq homotopy on $X \to Y$. Then $\Gamma(h)^*$ is Hausdorff and paracompact.*

For (a): Suppose $p_1(h)$ is ApS. The compactness of I guarantees that if $\{N(y_0)\}$ is an open base at y_0, then $\{N(y_0) \times I\}$ is an open base for $y_0 \times I$. Hence for an arbitrary $N(y_0)$, $N(y_0) \times I \supset N(y_0, I) \supset N'(y_0) \times I$ which yields the $ApSw$ condition. If $p_1(h)$ is $ApSw$, let $N(y_0)$ and $N'(y_0)$ be an associated pair and define

$$M(y_0) = \{(s, y) \mid \text{for some } x \in S_p,\ y \in h(x, s),\ h(x, s) \cap N'(y_0) = \emptyset,$$
$$h(x, s) \cap N(y_0) \neq \emptyset,\ h(x, s) \cap N(y_0)^{\sim} \neq \emptyset\} \subset I \times Y.$$

$C(y_0) = M(y_0) \cap (I \times N(y_0))$. Suppose $z = (\bar{y}, \bar{s})$ is a cluster point of $C(y_0)$ in $I \times N(y_0)$. If a cofinal collection of neighborhoods of $(\bar{y}, \bar{s})$ excludes all but y, s points in $M(y_0)$ which have x antecedents in a fixed finite subset π of S_p, then $z \in C(y_0)$, since the graph of h restricted to $\pi \times I$ is closed in $Y \times I$. If every $U(z)$ in a cofinal collection of neighborhoods of $(\bar{y}, \bar{s})$ contains points of $M(y_0)$ with a nonfinite set of x antecedents, this would contradict the $ApSw$ property for $\bar{y}$ (compare 2.7). Hence $C(y_0)$ is closed in $I \times N(y_0)$ so $N(y_0, I)$ can be defined as $I \times N(y_0) - C(y_0)$.

For (b): Following Lemma 2.1, $B(h)$ can be shown to be closed in $X \times I \times Y \times \Pi_{S_p}$ and so is paracompact. We show g is closed. Thus, suppose C is closed in $B(h)$. If $c \in C$, add the roof $R(x, s)$ containing c, and so obtain C^+. Suppose C^+ were not closed in $B(h)$. The nontrivial cases of possible cluster points *not* in C^+ are

(i) $b_0 = (x_0, s_0, y_0, t1_{x0}),\ x_0 \in S_p,\ 0 < t < 1;$

(ii) $b_0 = x_0, s_0, y_0, 1_{x0},$

(iii) $b_0 = x_0, s_0, y_0\ *,\ x_0 \in S_p.$

We dispose of (i) by separating b_0 from C by an open set containing no roof points, viz by the Cartesian product of open sets

$$O(b_0) = (V(x_0) \times A(s_0) \times W(y_0) \times U) \cap B$$

where $U = \{\psi \mid t/2 < \psi(x_0) < 1\}$. Since B selects those points in U for which $\psi(x) = 0$, $x \neq x_0$, $O(b_0) \cap C^+ = \varnothing$. Case (ii) is settled by noting that no roof point on $R(x_0, s_0)$ is a cluster point for $\{R(x, s) \mid x \neq x_0,\ s \in I\}$. Hence (ii) can arise only if b_0 is a cluster point of C^+ points on the roofs $R(x_0, s)$ for $x = x_0$. The fact that b maps compact sets onto compact sets implies that a point of C must have been on $R(x_0, s_0)$ and so $b_0 \in C^+$. For (iii), b_0 is the cluster point of roof points in C^+, $r_a = x_a, s_a, y_a, 1_{x_a}$, $a \in A$. By $ApSw$ there is an $N'(y_0)$ such that $b(x_a, s_a) \cap N'(y_0) \neq \varnothing$ implies $b(x_a, s_a) \subset N(y_0)$ except for a finite exceptional subset of S_p. Since y_0 is a cluster point, $N'(y_0)$ contains a cofinal collection $\{y_a\}$ where y_a is the coordinate of r_a and hence $N(y_0) \times U(*)$ contains almost all the roofs R_a associated with these y_a values. For each of these roofs pick c_a a point of C on this roof. Then b_0 is a cluster point of C and hence is in C. Therefore $gC^+ = gC$ is closed in $\Gamma(b)^*$.

Since cluster points of vertices in $\Gamma^*(b)$ are in $X \times I \times Y \times *$, for T_2 separation one need show only that the points $z_1 = (x_0, s_0, y_1, *)$ and $z_2 = (x_0, s_0, y_2, *)$ can be separated where $y_1 \cup y_2 \in b(x_0, s_0)$. (All other point pairs can obviously be separated.) Thus let $N_1(y_1) \cap N_2(y_2) = \varnothing$. Pick $N(y_1, I) \subset N(y_1) \times I$ and $N(y_2, I) \subset N(y_2) \times I$ and let π be the finite subset of S_p for which for some $t \in I$ and, for $j = 1$ or 2,

$$b(x, t) \cap N(y_j, t) \neq \varnothing, \qquad b(x, t) \cap N(y_j, t)^{\sim} \neq \varnothing.$$

Let U be defined by $\{\psi \mid \psi(x) < 1/2,\ x \in \pi\}$. Then

$$O_j = B(b) \cap (X \times N(y_j, I) \times U), \qquad j = 1, 2,$$

are separating neighborhoods in $B(b)$. Since no partial roofs enter, $g^{-1}gO_j = O_j$ whence gO_1 and gO_2 are separating open sets.

Our homotopy developments must circumvent various obstacles. Thus almost no general statement is valid for the composition of acyclic or ApS transformations but we need only that b ApS and e inclusion yields be ApS. We illustrate the necessity of an upper bound for p in S_p and ApS in usc maps of spheres.

Example. The mv homotopy $b: S^n \times I \to S^n$ is defined by: b is 1, the identity, for $s = 0$ and shrinks S^n uniformly on arcs through two poles to one of the poles. The singular set S_n consists of the other pole and 1 is n-homotopic to the constant mapping yet $h_0^*(n) \neq h_1^*(n)$ (Theorem 10.7 below implies only $h_0^*(m) = h_1^*(m)$ for $m > n+1$).

The next theorem is basic.

Theorem 10.7. *If* $f \sim_{pq} g$, $q \geq p+2$, *and* h *describes the homotopy, then* $h(m)^*$ *exists and* $f^*(m) = g^*(m)$ *for* $p+2 \leq m < q$.

The proof is reminiscent in part of that of Theorem 3.1, but there are deviations occasioned by the possibility of lines of singularities. Accordingly, the lower bounds for m are higher than might at first have been conjectured.

The map $P_1\colon \Gamma(h)^* \to X \times I$ is defined by

$$P_1(x \times s \times h(x, s)^*) = x \times s \qquad \text{for } x \in S_p,$$

$$P_1(x \times s \times h(x, s) \times *) = p_1(x \times s \times h(x, s)) = x \times s \quad \text{for } x \notin S_p.$$

We shall often use $h(x, s)$ for $h(x, s) \times *$ where the context makes the situation clear.

We show first that p_1 is closed. If not, suppose C is closed in $\Gamma(h)$ but p_1C is not closed in $X \times I$. Write z for (x, s). Hence some point $z_0 \notin p_1C$ is a cluster point of p_1C. By the local compactness of $X \times I$, there is a compact neighborhood U of z_0. The graph of $h \mid U$ is closed in $U \times h(U)$ and hence $p_1^{-1}U = \Gamma(h \mid U)$ is compact. Accordingly, $p^{-1}U \cap C$ is compact and nonempty. Choose a fundamental system of compact neighborhoods U_α of z_0. By the finite intersection property,

$$\emptyset \neq \bigcap p_1^{-1}U_\alpha \cap C = p_1^{-1} \bigcap U_\alpha \cap C = p_1^{-1}z_0 \cap C,$$

a contradiction. Furthermore the type of argument used in Theorem 3.1 to show F is closed demonstrates that P_1 too is closed.

A critical assertion is that

$$(10.71) \qquad W(x_0)^* = x_0 \times \bigcup_{s \in I} s \times (h(x_0, s)^* - h(x_0, s) \times *), \qquad x_0 \in S_p,$$

is an open set in $\Gamma(h)^*$. To verify this let $V(x_0)$ be open in X so that $V(x_0) \times I$ is open in $X \times I$. Let $\{\psi \mid \psi(x_0) > 0\}$ be open in $\Pi_{S_p} I(x)$. Evidently, $W(x_0) = p_1^{-1}(V(x_0) \times I) \times \{\psi \mid \psi(x_0) > 0\} \cap B(h)$ is open in $B(h)$. Since no partial roofs enter, it is easily seen that $g^{-1}gW(x_0) = W(x_0)$. Hence $gW(w_0)$ is open in $\Gamma(h)^*$. Moreover, it may be verified that $gW(x_0) = W(x_0)^*$. Thus

$$(10.72) \qquad \Gamma(h)^* - \Gamma(h) = \bigcup_{x \in S_p} W(x)^*$$

is a union of disjunct open sets. Hence by Lemma 2.6,

$$(10.73) \qquad H^m_{\Phi|\Gamma(h)^* - \Gamma(h)}(\Gamma(h)^* - \Gamma(h)) = \prod_{S_p} H^m_{\Phi|W(x)^*} W(x)^*.$$

Write

$$A(x) = x \times \bigcup_{s \in I} s \times h(x, s)^*, \qquad C(x) = x \times \bigcup_{s \in I} s \times h(x, s) \times *.$$

By excision, for Alexander cohomology,

$$H^m_{\phi | W(x)^*} W(x)^* \approx H^m(A(x), C(x)). \tag{10.74}$$

Note that $x \times s \times h(x, s)^* = p_1^{-1}(x \times s)$, $x \in S_p$, and that

$$P_1 \; x \times \bigcup_{s \in I} s \times h(x, s)^* = x \times I.$$

Since $h^*(x, s)$ is acyclic by construction for each s the standard Vietoris-Begle theorem guarantees that the cohomology groups of $x \times \bigcup_{s \in I} (s \times h^*(x, s))$ are those of $x \times I$ and are therefore trivial.

The cohomology sequence for the pair $A(x)$, $C(x)$ therefore yields, in view of (10.74), the isomorphism

$$H^m C(x) \approx H^{m+1}_{\phi | W(x)^*} W(x)^* \tag{10.75}$$

and by (10.73) and (10.74)

$$H^{m+1}(\Gamma(h)^*, \Gamma(h)) \approx \prod_{S_p} H^m C(x). \tag{10.76}$$

We need to examine the map $p_x \colon C(x) \to x \times I$, $x \in S_p$, in detail. For some, if not all $s \in I$, $p_1^{-1}(x \times s)$ is nonacyclic in some grade inferior to p.

Let d_r be the dimension of $\sigma_r = \{s \mid H^r(x \times s \times h(x, s)) \not\approx 0\}$. Evidently, an upper bound for d_r is obtained when this set is I so max $d_r = 1$. Introduce

$$n = 1 + \max_{r < q+1;\, \sigma_r \neq \emptyset} (d_r + r).$$

By the ApS property

$$n = 1 + (1 + p - 1) = p + 1. \tag{10.77}$$

We must now draw on results in [5] that imply for

$$H^m C(x) \xleftarrow{\;p_x(m)^*\;} H^m(x \times I) = 0, \qquad x \in S_p,$$

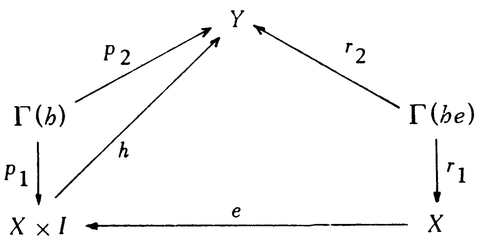
that $p_x(m)^*$ is a $p+1$, $q+1$ morphism. However, since $p_x(p+1)^*$ is an epimorphism onto the trivial group, we can assert that $p_x(m)^*$ is an isomorphism for $p + 1 \leq m \leq q$.

The exact cohomology sequence reduces to

$$\begin{array}{ccccccc} \prod_{S_p} H^m(C(x)) & \to & H^{m+1}\Gamma(h)^* & \xrightarrow{\alpha(m+1)} & H^{m+1}(\Gamma(h)) & \to & \prod_{S_p} H^{m+1}C(x). \\ & & \nwarrow{\scriptstyle P_1(m+1)^*} & & \nearrow{\scriptstyle p_1(m+1)^*} & & \\ & & & H^{m+1}(X \times I) & & & \end{array} \tag{10.78}$$

Since $H^m(C(x))$ is trivial for $p+1 \leq m \leq q$, $\alpha(m+1)$ is an isomorphism, for $p+1 \leq m \leq q-1$. By the conventional Vietoris-Begle theorem, $P_1^*(m+1)$ is an isomorphism for $0 \leq m \leq q-1$.

The commutativity of the triangle in (10.78) yields $p_1(m)^*$ is an isomorphism for $p+2 \leq m \leq q$. We turn to Lemma 10.1 and consider the special case that Y is $X \times I$ and Z is Y. The diagram



is commutative where e is either $e(0)$ or $e(1)$, viz $e(s)$: $x \longrightarrow x \times s$. Then comparison with the diagram of Lemma 10.1 with the obvious identifications yields existence of $h^*(m)$ and

$$(e^*(0)h^*)(m) = f^*(m), \qquad (e^*(1)h^*)(m) = g^*(m).$$

It is well known, however, [4, p. 178] that $e(s)^*$ is independent of s for all m. Hence

$$f^*(m) = g^*(m), \qquad p+1 < m \leq q.$$

Just as in Lemma 5.2 the ApS property for usc maps and compact spaces is equivalent to the condition that, *for every vicinity* V *of the diagonal* Δ in $X \times X$ and *for almost all* $x \in S_p$, $f(x) \times f(x) \subset V$. The proof is a trivial modification of the proof of Lemma 5.2. Moreover for compacta these properties are again easily seen to be equivalent to the condition

(C) S_p is denumerable and, for arbitrary $\epsilon > 0$, for at most a finite subset of S_p, $\operatorname{diam} f(x) > \epsilon$.

The following lemma extends to nonacyclic usc transformations, results known for continuous maps.

Lemma 10.8. (a) *No set valued usc transformation f exists of the $n+1$ disk D^{n+1} onto S^n where $f(x)$ is disjunct from $-x$, the singular subset is S_{n-2},* (C) *is satisfied and $f(x)$ is acyclic in dimensions $n-2$, $n-1$ and n.* (b) *If $f(x) = x$ on the boundary, S_{n-2} can be replaced by S_{n-1}.*

The proof is by contradiction. Suppose i is the inclusion map on S^n to D^{n+1}. Denote by g the composite fi. Then by Lemma 10.1 and Theorem 10.7

(10.81) $$i^*(n)f^*(n) = g^*(n).$$

We give the verification in more detail for (b) so 10.7 can be replaced by 3.1. Thus take $X = D^{n+1}$, $Y = S^n$ and $p_1^{-1}x = x \times f(x) \in \Gamma(f)$. By an argument used before, p_1 is closed. The hypotheses of 10.8(b) imply p_1 is $A\, n-1\, S$. Accordingly by 3.1 $p_1^*(n)$ is an isomorphism so by (10a), $f^*(n)$ exists, whence there results (10.81).

However g is $n-2$ homotopic to the identity map 1. To see this, note that by compactness

$$\inf_{x \in S} d(-x, f(x)) = \epsilon > 0. \tag{10.82}$$

Let (a, b) be barycentric coordinates of a point on a great semicircle $T(x)$ through x and $-x$, i.e. $0 \le a \le 1$, $(a + b = 1)$. Define $h(x, (a, b), s)$ as the point on $T(x)$ with coordinates $s + (1 - s)a$, $(1 - s)b$, $s \in I$. With the restriction of $z = (a, b) \in T(x)$ to points distant at least ϵ from $-x$, the continuity of h in x, z, s is an easy consequence of a triangle inequality argument. We prove that $h'(x, s) = h(x, g(x), s)$ is usc. Let $x^n \to \bar{x}$, $s^n \to \bar{s}$ and $w^n \to \bar{w}$ where $w^n \in h'(x^n, s^n)$. For some $z^n \in g(x^n)$, $w^n = h(x^n, z^n, s^n)$. By compactness, at worst a subsequence, again denoted by $\{z^n\}$, converges to $\bar{z}$. Since g is usc, $\bar{z} \in g(\bar{x})$. The continuity of h then insures the desired $\bar{w} = h(\bar{x}, \bar{z}, \bar{s}) \subset h'(\bar{x}, \bar{s})$.

For $s < 1$, $h(x, g(x), s)$ is a homeomorph of $g(x)$. Hence $p_1(h')$, cf. Definition 10.4, is $A(n-2)S$ and accordingly h' defines an *mv* $n-2$ homotopy of g to 1, the identity map. Hence for $m = n$, the right-hand side of (10.81) is nonzero while i^* is plainly 0.

In line with the considerations in this section we extend the notion of degree.

Definition 10.9. Let f be a set valued usc map of $S^n \to S^n$ which satisfies (C) for S_{n-2} and is acyclic in dimensions $n-2, n-1$, and n. Let γ^n be a generator of S^n. Assume the constant sheaf J. Then $f^*(n)$ is defined and deg f *is the integer* d in $f^*\gamma^n = d\gamma^n$.

The standard simple properties follow and need the full force of 10.7. For instance

Lemma 10.10 (a) *If* $f \sim_{n-2} g$ *where*

$$S^n \overset{f}{\underset{g}{\rightrightarrows}} S^n$$

then deg f = deg g.

(b) *If f satisfies* (C) *and is acyclic in dimensions* $n-2, n-1$, *and* n *and if* deg $f \neq 0$, *then f maps* S^n *onto* S^n.

The usual retraction to a point argument goes through. Further properties and extensions will be given elsewhere.

We shall refer to x where $x \in f(x)$ as a *fixed point*. The following result is perhaps the most striking contribution of the paper. It formulates a new type of fixed point theorem marking the first extension of the classical Eilenberg-Montgomery theorem [14] to nonacyclic images.

Theorem 10.11. *Let f be a usc self map of the closed n simplex* D^n, $n > 3$.

Let S_{n-2} be the singular subset defined by the condition that $f(x)$ is a convex set or more generally star convex with respect to x for $x \notin S_{n-2}$ and, for $x \in S_{n-2}$, $f(x)$ is a finite union of convex sets of which at most $n-1$ are of dimension greater than $n-3$. Require also that (C) *be satisfied. Then f has a fixed point.*

We shall make tacit of the fact that $H_m(F(x), Q)$ is obviously finitely generated and is therefore isomorphic to its conjugate vector space $H^m(F(x), Q)$.

Suppose no x satisfies $f(x) \supset x$. For each $x \in D^n$ denote by $f'(x)$ the reflection of $f(x)$ in x. Let $C(x)$ be the infinite cone of vertex x and base $f'(x)$. Let $F(x) = \dot{D}^n \cap C(x)$ where $\dot{D}^n$, a homeomorph of S^n, is the boundary of D^n. Note that when $x \in \dot{D}^n$, this construction yields $F(x) = x$. Under the hypotheses of the theorem $F(x)$ is acyclic for $x \notin S_{n-2}$. We show that $F(x)$ is $n-2$ acyclic. For $x \in S_{n-2}$, the worst case is that when $n-1$ convex sets of dimension $n-2$, $n-1$, or n occur and these sets each contain an $n-2$ simplex σ^i_{n-2} where $\{\sigma^i_{n-2}, i = 1, \cdots, n-1\}$ constitute the partial faces of an $n-1$ simplex σ_{n-1}. Under F, either σ_{n-1} goes into an $n-1$ simplex on $\dot{D}^n$, in which case the set $\{\sigma^i_{n-2}\}$ goes into a corresponding set of simplexes and so does not carry a nonbounding $n-2$ cycle, or σ_{n-1} goes into an $n-2$ simplex $\tilde{\sigma}_{n-2}$, whence the $n-2$ faces go into a simplicial subdivision, and by the acyclicity of $|\sigma_{n-2}|$ the image by F is again $n-2$ acyclic. Similarly, $F(x)$ is $n-1$ and n-acyclic.

Finally, $F(x)$ is usc. First note $F(x)$ is closed since $f(x)$ and hence $f'(x)$ also is compact and so is $C(x) \cap D^n = \check{C}(x)$ whence $F(x) = \check{C}(x) \cap S^{n-1}$ is compact. From the assumption that no fixed point exists, it follows that

$$\inf d(x, f(x)) = \delta > 0. \tag{10.11a}$$

Otherwise, for proper choice of $\{x^n\}$, $d(x^n, f(x^n)) < 1/n$ whence by compactness $x^n \to \bar{x}$, $y^n \to \bar{y}$ where $y^n \in f(x^n)$. Since f is usc, $\bar{y} \in f(\bar{x})$. In short, $d(\bar{x}, f(\bar{x})) = 0$ and since $f(\bar{x})$ is closed, $\bar{x} \in f(\bar{x})$. Let $x^n \to \tilde{x}$ and let $z^n \in F(x^n)$. By compactness a subsequence of x^n and z^n can be taken (though here and later the notation is not changed) so that $(x^n, z^n) \to \tilde{x}, \tilde{z}$. We need establish that $\tilde{z} \in F(\tilde{x})$. Each z^n is the end point on $\dot{D}^n$ of a segment through $[x^n, y^n]$ where $y^n \in f'(x^n)$. By compactness again we may choose subsequences, if necessary, for which $(x^n, y^n) \to \tilde{x}, y_0$. Since f is usc, $y_0 \in f'(\tilde{x})$. Define z_0 as the point on $\dot{D}^n$ along $\tilde{x}, y_0$. We claim $z_0 = \tilde{z}$. Otherwise $d(z_0, \tilde{z}) = \nu > 0$ and a spherical neighborhood $M(z_0)$ of radius $\nu/4$ would contain no cofinal set of z^n's. For simplicity of exposition assume first that $\tilde{x} \notin \dot{D}^n$. Let $N(\tilde{x})$ and $N'(y_0)$ be spherical neighborhoods of radius $\delta/2^k$ where δ is the constant above. Suppose diam $D^n = 2$. By similar triangles it is easy to see $d(z_0, z^n) < (\delta/2^k)/(\delta/2) = 2^{1-k}$. Hence for k sufficiently large $2^{1-k} < \nu/4$ and

$M(z_0)$ would contain a cofinal subset of $\{z^n\}$, a contradiction. Accordingly, $\tilde{z} = z_0$ and F is usc.

Plainly, the argument just given is unaffected if $\tilde{x} \in \dot{D}^n$. Accordingly F is a usc map of D^n into $\dot{D}^n$ and by a homeomorphism it may be viewed as a map of the Euclidean n disk into the Euclidean $n-1$ sphere.

To see that (C) is satisfied for F note first that the singular set for $F(x)$ is contained in that for $f(x)$. Moreover, diam $F(x) \leq c$ diam $f(x)$ where $c = 2/\delta$.

Therefore, since $F(x) = x$ on the sphere, the conditions of Lemma 10.8b are satisfied and the assumption of no fixed point has led to a contradiction with the conclusion of that lemma.

We shall show in another place that inter alia the convexity conditions in (10.11) can be eliminated by use of a homotopy theorem of the type of 10.7 applied to $D^n \times I \to \dot{D}^n \times I$ based on a natural image of $f(x)$ in $C(x)$ (cf. Pacific J. Math.).

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